Exact Solutions for Thermal Problems

Buoyancy, Marangoni, Vibrational and Magnetic-Field-Controlled Flows

Marcello Lappa*1

Telespazio
Via Gianturco 31, 80046, Napoli, Italy
*1 marlappa@unina.it

Abstract

In general, the thermal-convection (Navier-Stokes and energy) equations are nonlinear partial differential equations that in most cases require the use of complex algorithms in combination with opportune discretization techniques for obtaining reliable numerical solutions. There are some cases, however, in which such equations admit analytical solutions. Such exact solutions have enjoyed a widespread use in the literature as basic states for determining the linear stability limits in some idealized situations. This review article provides a synthetic review of such analytical expressions for a variety of situations of interest in materials science (especially crystal growth and related disciplines), including thermogravitational (buoyancy), thermocapillary (Marangoni), thermovibrational convection as well as "mixed" cases and flow controlled via static and uniform magnetic fields.

Keywords

Thermal Convection; Navier-Stokes Equations; Analytic Solutions

Introduction

Finding solutions to the Navier-Stokes equations is extremely challenging. In fact, only a handful of exact solutions are known. For the cases of interest in the present article (buoyancy, Marangoni, vibrational and magnetic convection), in particular, these exact solutions exist when the considered system is infinitely extended along the direction of the imposed temperature gradient or the orthogonal direction; in general, such solutions are regarded as reasonable approximations in the steady state of the flow occurring in the core of real configurations which are sufficiently elongated (the core is the region sufficiently away from the end regions, where the fluid turns around, to be considered not influenced by such edge effects). Even though limited to cases of great simpicity, these solutions have proven able to yield directly or indirectly insights and understanding that would have been difficult to obtain otherwise.

Hereafter, first the attention is focused on the case of systems subject to horizontal heating (or to temperature gradient along the direction in which they are assumed to be infinitely extended), then other possible variants and configurations are considered in the section on "inclined systems" and subsequent ones.

Analytic Solutions for Thermogravitational and Marangoni Flows

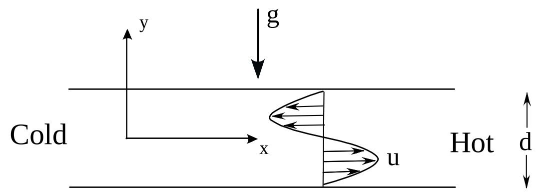


FIG. 1 SKETCH OF LAYER OF INFINITE EXTENT SUBJECT TO HORIZONTAL HEATING

In the following three subsections (focusing on the Hadley flow, Marangoni flow and related hybrid states) the horizontal boundaries are assumed to be located at y=-1/2 and 1/2, respectively. The velocity components along y and z are zero (v=w=0) and the component along x is solely a function of y, i.e. u=u(y) (which means the assumption of "plane-parallel flow" is considered). Temperature depends on x and y (i.e. these solutions are essentially twodimensional). Rayleigh and Marangoni numbers are defined as $Ra=GrPr=g\beta_T\gamma d^4/\nu\alpha$ and $Ma= RePr=\sigma_T\gamma d^2/\mu\alpha$, respectively (where γ is a rate of uniform temperature increase along the x axis, d is the distance between the boundaries, α is the thermal diffusivity, ν is the kinematic viscosity, β_T is the thermal expansion coefficient, Pr the ratio between ν and α, Gr and Re the Grashof and Reynolds numbers, respectively). Referring velocity and temperature to the scales α/d and γd, respectively and scaling all distances on d, the governing equations in nondimensional form (incompressible form) read

$$\underline{\nabla}\cdot\underline{V}=0 \tag{1}$$

$$\frac{\partial\underline{V}}{\partial t}=-\underline{\nabla}p-\underline{\nabla}\cdot\left[\underline{V}\underline{V}\right]+\Pr\nabla^2\underline{V}-\Pr Ra\ T\ \underline{i}_g \tag{2}$$

$$\frac{\partial T}{\partial t}+\underline{\nabla}\cdot\left[\underline{V}T\right]=\nabla^2 T \tag{3}$$

where $\underline{i}_g$ is the unit vector along the direction of gravity and the Boussinesq approximation has been used for the buoyancy production term in the momentum equation.

In the presence of a free liquid/gas interface such equations must be considered together with the socalled Marangoni boundary condition, which neglecting viscous stress in the gas (here the dynamic viscosity of the gas surrounding the free liquid surface will be assumed to be negligible with respect to the viscosity of the considered liquid) reads

$$\frac{\partial \underline{V}_S}{\partial n} = -Ma\ \underline{\nabla}_S T \tag{4}$$

where n is the direction locally perpendicular to the considered elementary portion of the free interface, $\underline{\nabla}_s$ the derivative tangential to the interface and $\underline{V}_s$ the surface velocity vector.

For the case considered here (steady flow with u=u(y) and v=w=0, as shown in Fig. 1) eqs. (1)-(3) reduce to :

$$\frac{\partial p}{\partial x} = \Pr \frac{\partial^2 u}{\partial y^2} \tag{5}$$

$$\frac{\partial p}{\partial y} = \Pr RaT \tag{6}$$

$$u\frac{\partial T}{\partial x} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \tag{7}$$

while eq. (4) can be rewritten as:

$$\frac{\partial u}{\partial y} = -Ma\frac{\partial T}{\partial x} \tag{8}$$

Assuming solutions in the general form (with Ma=0 or Ra=0 for pure buoyancy or Marangoni flows, respectively):

$$\underline{V} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} Ra\ \ g_1(y) + Ma\ \ g_2(y) \\ 0 \\ 0 \end{bmatrix} \tag{9}$$

$$T = x + Ra\ \ f_1(y) + Ma\ \ f_2(y) \tag{10}$$

f_1, f_2, g_1 and g_2 can be determined as polynomial expressions with constant coefficients satisfying specific equations obtained by substituting eqs. (9) and (10) into eq. (7) and into the equation resulting from cross-differentiation of eqs. (5) and (6) (i.e. $\frac{d^3u}{dy^3} = Ra\frac{\partial T}{\partial x}$).

Such system of equations read:

$$g_1 = \frac{d^2 f_1}{dy^2} \tag{11a}$$

$$g_2 = \frac{d^2 f_2}{dy^2} \tag{11b}$$

$$Ra\left(\frac{d^3 g_1}{dy^3} - 1\right) + Ma\frac{d^3 g_2}{dy^3} = 0 \tag{12}$$

to be supplemented with the proper boundary conditions:

Kinematic conditions:

solid boundary: u=0 $\rightarrow$ $g_1(y)=g_2(y)=0$ (13)

free surface: eq. (8) $\rightarrow \frac{dg_1}{dy} = 0$ and $\frac{dg_2}{dy} = -1$ (14)

Thermal conditions:

Adiabatic boundary: $\frac{\partial T}{\partial y} = 0 \rightarrow \frac{df_1}{dy} = \frac{df_2}{dy} = 0$ (15)

Conducting boundary: T=x$\rightarrow$ $f_1=f_2=0$ (16)

In addition, it must be taken into account that since the considered parallel flow is intended to model a slot with distant end walls, continuity requires that the net flux of fluid at any cross section of the slot be zero), i.e.:

$$\int_{-1/2}^{1/2} u dy = 0 \rightarrow \int_{-1/2}^{1/2} g_1 dy = \int_{-1/2}^{1/2} g_2 dy = 0 \tag{17}$$

Before going further with the description of solutions satisfying the system of equations (11-17), it is worth noting that some insights into their expected polynomial order can be given immediately on the basis of eq. (12). According to such equation, in fact, $d^3g_1/dy^3=1$ for Ma=0, while $d^3g_2/dy^3=0$ for Ra=0, which leads to the general conclusion that the polynominal expression for u will be of the third order for pure buoyancy flow and of the second order for pure Marangoni flow, while (on the basis of eqs. (11)) the respective temperature profiles are of the fifth and forth orders in y.

Thermogravitational Convection: The Hadley flow

In line with the considerations above, in the case of liquid confined between two horizontal infinite walls with perpendicular gravity, eqs. (5)-(7) admit as an exact solution the following velocity profile:

$$u = \frac{Ra}{6}\left(y^3 - \frac{y}{4}\right) \tag{18}$$

generally known as the Hadley flow (Hadley, 1735) as it was originally used as a model of atmospheric circulation (see, e.g., Lappa, 2012) between the poles and equator (from a technological point of view, this

solution is also strongly relevant to the manufacture of bulk semiconductor crystals).

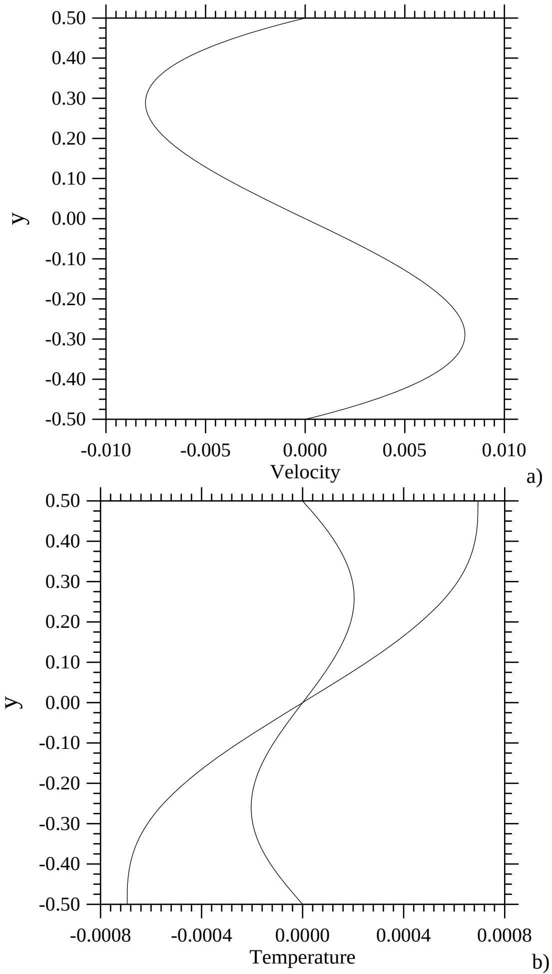
The corresponding temperature profile changes according to the type of boundary conditions considered. For adiabatic walls it reads:

$$T = x + \frac{Ra}{120} y\left(y^4 - \frac{5}{6}y^2 + \frac{5}{16}\right) \quad (19a)$$

whereas for conducting boundaries it becomes:

$$T = x + \frac{Ra}{120} y\left(y^4 - \frac{5}{6}y^2 + \frac{7}{48}\right) \quad (19b)$$

The polynomial expressions f and g for such solutions are plotted in Figs. 2.



FIGS. 2 EXACT SOLUTION FOR THE CASE OF THERMOGRAVITATIONAL CONVECTION IN INFINITE LAYER WITH TOP AND BOTTOM SOLID WALLS: (a) VELOCITY PROFILE g(y); (b) TEMPERATURE PROFILE f(y) FOR ADIABATIC (SOLID) AND CONDUCTING (DASHED) BOUNDARIES.

Marangoni Flow

In the absence of gravity and replacing the upper solid wall with a liquid-gas interface supporting the development of surface-tension-driven (Marangoni) convection, the exact solution reads (Birikh, 1966):

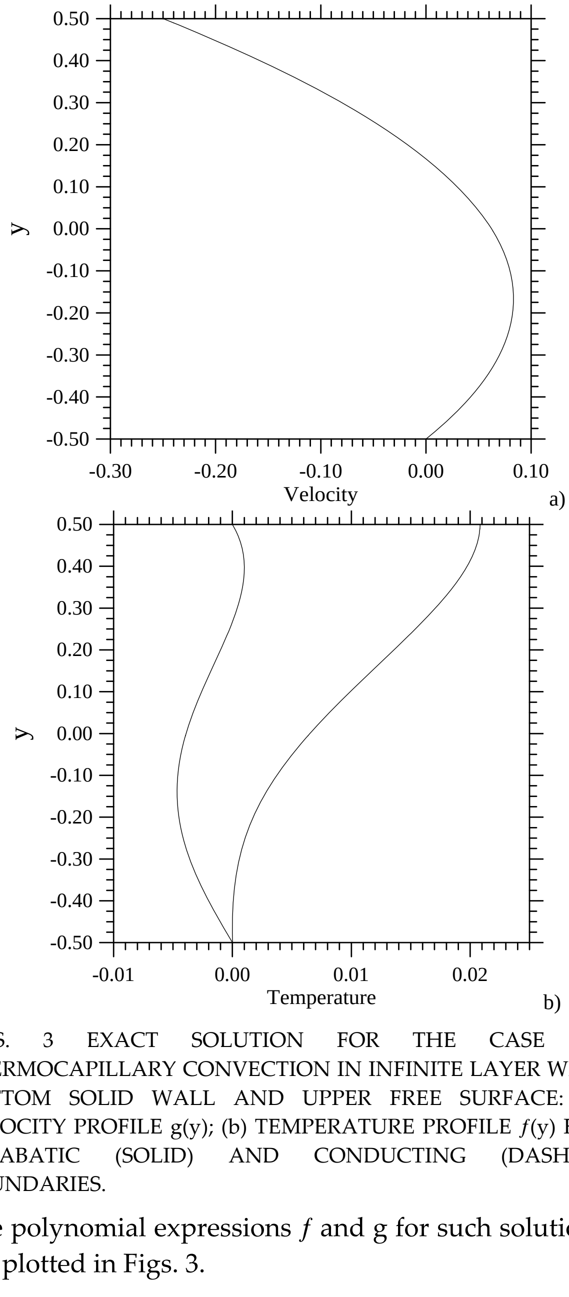
$$u = -\frac{Ma}{4}\left(3y^2 + y - \frac{1}{4}\right) \quad (20)$$

For adiabatic interface and insulated bottom wall the temperature distribution can be expressed as:

$$T = x - \frac{Ma}{48}\left(3y^4 + 2y^3 - \frac{3}{2}y^2 - \frac{3}{2}y - \frac{5}{16}\right) \quad (21a)$$

that for conducting boundaries must be replaced with:

$$T = x - \frac{Ma}{48}\left(3y^4 + 2y^3 - \frac{3}{2}y^2 - \frac{1}{2}y + \frac{3}{16}\right) \quad (21b)$$



FIGS. 3 EXACT SOLUTION FOR THE CASE OF THERMOCAPILLARY CONVECTION IN INFINITE LAYER WITH BOTTOM SOLID WALL AND UPPER FREE SURFACE: (a) VELOCITY PROFILE g(y); (b) TEMPERATURE PROFILE f(y) FOR ADIABATIC (SOLID) AND CONDUCTING (DASHED) BOUNDARIES.

The polynomial expressions f and g for such solutions are plotted in Figs. 3.

Hybrid Buoyancy-Marangoni States

As shown by eqs. (5)-(7), under the considered conditions ($u \neq 0$, $v=w=0$), the nonlinear convective term of the momentum equation becomes zero and the

energy equation reduces to $u=\partial^2T/\partial y^2$, i.e. the governing equations are linear; as a consequence, more complex solutions can be built as superposition (addition) of other simple existing solutions.

Along these lines, in the presence of vertical gravity and a liquid-gas interface supporting the development of surface-tension-driven (Marangoni) convection, the velocity profile can be obtained as the sum of two terms, the first term corresponding to the pure buoyancy-driven flow and the second term representing the contribution of a pure thermocapillary-driven flow, which provides a simple theoretical explanation for the general form given by eqs. (9) and (10).

The resulting shear flow is set up by a combined effect of buoyancy and viscous surface stress due to the temperature dependence of surface tension. The first contribution related to pure buoyancy flow, however, exhibits some differences with respect to eq. (18) as the upper solid wall considered earlier must be replaced with a stress-free boundary (see Fig. 4); the velocity profile reads:

$$u=\frac{Ra}{48}\left(8y^3-3y^2-3y+\frac{1}{4}\right) \tag{22}$$

Hence, the resulting profile for mixed gravitational-Marangoni convection has the form:

$$u=\frac{Ra}{48}\left(8y^3-3y^2-3y+\frac{1}{4}\right)-\frac{Ma}{4}\left(3y^2+y-\frac{1}{4}\right) \tag{23}$$

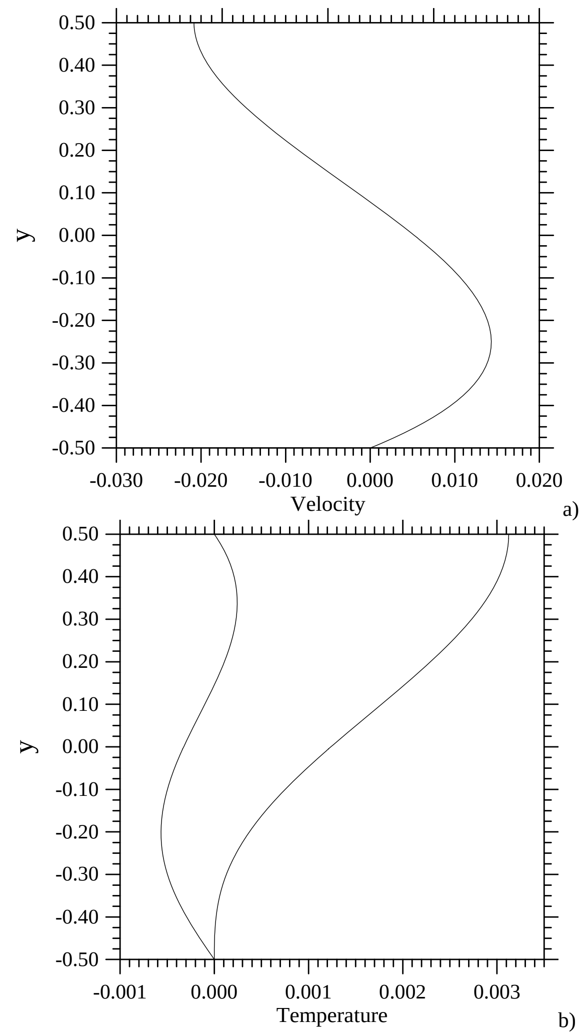
The associated temperature distribution is:

$$T=x+\frac{1}{48}\left\{\frac{Ra}{20}\left[8y^5-5y^4-10y^3+\frac{5}{2}y^2+5y+\frac{19}{16}-3\eta\left(y+\frac{1}{2}\right)\right]-Ma\left[3y^4+2y^3-\frac{3}{2}y^2-\frac{3}{2}y-\frac{5}{16}+\eta\left(y+\frac{1}{2}\right)\right]\right\} \tag{24}$$

The cases of adiabatic and conducting horizontal surfaces can be obtained by setting in this relation $\eta=0$ and $\eta=1$, respectively.

Equation (24) can be extended to the more general case in which the bottom is conducting and a Biot number is introduced to describe heat transfer at the top free surface by assuming $\eta=Bi/(1+Bi)$. The solutions for symmetrical (top and bottom) conditions can again be recovered as limiting cases when $Bi\to 0$ and $Bi\to\infty$:

When $Bi\to\infty$, in fact, $\eta\to 1$ and eq. (24) reduces to the conducting case; when $Bi\to 0$ ($\eta\to 0$), it reduces to the insulating boundary conditions for the temperature profile.



FIGS. 4 EXACT SOLUTION FOR THE CASE OF PURE THERMOGRAVITATIONAL CONVECTION IN INFINITE LAYER WITH BOTTOM SOLID WALL AND UPPER STRESS-FREE SURFACE: a) VELOCITY PROFILE g(y); b) TEMPERATURE PROFILE f(y) FOR ADIABATIC (SOLID) AND CONDUCTING (DASHED) BOUNDARIES.

General Properties

The present section is devoted to illustrate some fundamental properties of the solutions introduced in the preceding sections with respect to the well-known Rayleigh's condition, which so much attention has enjoyed in the literature as a means for gaining insights into the fluid-dynamic instabilities of parallel flows in the limit as $Pr\to 0$ (Rayleigh, 1880; Lin, 1944; Rosenbluth and Simon, 1964; Drazin and Howard, 1966).

Let us recall that such a theorem reads*: In a shear flow a necessary condition for instability is that there must be a point of inflection in the velocity profile u=u(y), i.e. a point where $d^2u/dy^2=0$.*

The second derivative of solution (18) gives:

$$\frac{d^2u}{dy^2} = Ra\ y \tag{25}$$

that means the profile has an inflection point at midheight (y=0) and satisfies the Rayleigh's necessary condition.

For pure buoyancy flow with upper free surface, i.e. solution (22), the second derivative reads:

$$\frac{d^2u}{dy^2} = \frac{Ra}{48}\left(48y - 6\right) \tag{26}$$

that gives the inflection point at a sligtly different position (y=1/8).

The inflection point, however, is no longer present for the case of pure Marangoni flow since:

$$\frac{d^2u}{dy^2} = -\frac{3}{2}Ma \neq 0 \tag{27}$$

that means Marangoni flow solutions of the type given by eq. (20) do not satisfy the Rayleigh's necessary condition.

The most interesting case in this regard is, perhaps, given by the mixed state represented by eq. (23). In such a case:

$$\frac{d^2u}{dy^2} = \frac{Ra}{48}\left(48y - 6 - 72W\right) \tag{28}$$

that makes the location of the inflection point a linear function of the nondimensional parameter W=Ma/Ra:

$$y = \frac{12W + 1}{8} \tag{29}$$

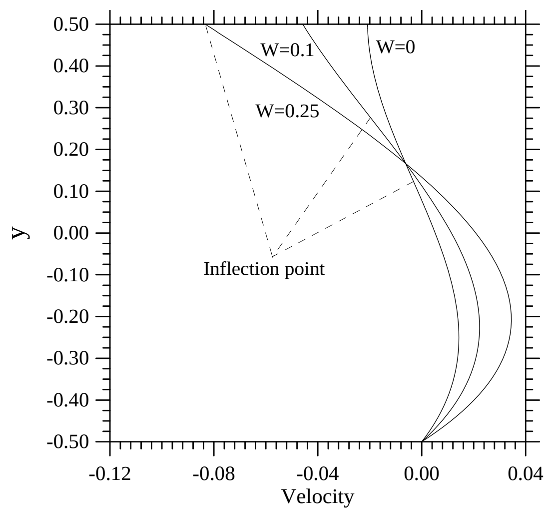


FIG. 5 VELOCITY PROFILE $g_1(y)+Wg_2(y)$ FOR THE CASE OF MIXED THERMOGRAVITATIONAL-THERMOCAPILLARY CONVECTION IN INFINITE LAYER WITH BOTTOM SOLID WALL AND UPPER STRESS-FREE SURFACE: THE INFLECTION POINT DISAPPEARS FOR W=0.25.

The inflection point disappears when y>1/2, i.e. for W>1/4. Accordingly, these solutions satisfy the Rayleigh's necessary condition only if 0<W<1/4 (see Fig. 5).

The Infinitely Long Liquid Bridge

For pure Marangoni flow, the equations of thermal-convection admit exact solutions also for the case of a liquid bridge assumed to be infinitely extended along the axial direction z (Xu and Davis, 1983). For such a case the solution is axisymmetric and reads:

$$w = \frac{Ma}{2}\left(r^2 - \frac{1}{2}\right) \tag{30}$$

$$T = -z - \frac{Ma}{32}\left(1 - r^2\right)^2 \tag{31}$$

The interface (adiabatic) is assumed to be located at r=1. The Marangoni number is defined as $Ma=RePr=\sigma_T \gamma R^2/\mu\alpha$ (where γ is the rate of constant temperature increase along the z axis, R is the radius of the liquid bridge). Velocity and temperature are referred to the scales α/R and γR, respectively (moreover, all distances are scaled on R).

According to eq. (30), the velocity component along the axis of the liquid bridge is solely a function of z (w=w(z)). Temperature is a function of both axial and radial coordinates (z and r).

Inclined Systems

The solutions given before for the case of a horizontal layer of fluid with no-slip walls subjected to a uniform temperature gradient along x can be extended to the more general case in which the layer is inclined by an angle 90°-θ with respect to the horizontal direction (see Fig. 6).

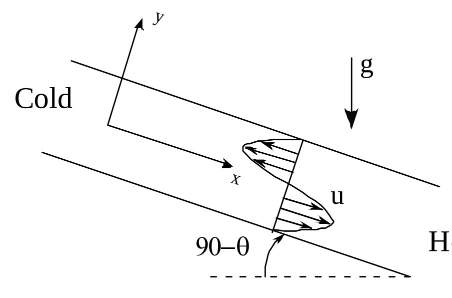


FIG. 6 SKETCH OF A LAYER OF INFINITE EXTENT INCLINED WITH RESPECT TO THE HORIZONTAL DIRECTION (TEMPERATURE GRADIENT ALONG THE x DIRECTION; θ ANGLE BETWEEN ∇T AND g).

In particular, the analytical solution for such a case has distinct expressions depending on the projection of the temperature gradient to the gravity vector, i.e. according to whether from a global point of view the layer tends to behave as a system heated from below or from above (depending on the sign of 90°-θ in Fig.

6), and depending on the type of thermal conditions along the walls.

In the following these expressions are given first for the case of adiabatic walls (eqs. (32)-(35)); then the configuration with conducting boundaries is considered (eqs. (38)-(41)).

Heating from below (0°<θ<90°)

$$u = \frac{Ra\sin(\theta)}{16}\left[2\frac{\sinh(2\xi y)\sin(\xi)-\sinh(\xi)\sin(2\xi y)}{\xi^2(\sinh(\xi)\cos(\xi)+\cosh(\xi)\sin(\xi))}\right] \quad (32)$$

$$T = x+\frac{1}{2}\tan(\theta)\left[2y-\frac{\sinh(2\xi y)\sin(\xi)+\sinh(\xi)\sin(2\xi y)}{\sinh(\xi)\cos(\xi)+\cosh(\xi)\sin(\xi)}\right] \quad (33)$$

Heating from above (90°<θ<180°)

$$u(y) = \frac{Ra\sin(\theta)}{16}\left[\frac{\cosh(2\xi y)\sin(2\xi y)\sinh(\xi)\cos(\xi)}{\xi^3(\sinh(\xi)\cosh(\xi)+\sin(\xi)\cos(\xi))}-\frac{\sinh(2\xi y)\cos(2\xi y)\cosh(\xi)\sin(\xi)}{\xi^3(\sinh(\xi)\cosh(\xi)+\sin(\xi)\cos(\xi))}\right] \quad (34)$$

$$T = x+\frac{1}{2}\tan(\theta)\left[2y-\frac{\sinh(2\xi y)\cos(2\xi y)\sinh(\xi)\cos(\xi)}{\xi(\sinh(\xi)\cosh(\xi)+\sin(\xi)\cos(\xi))}-\frac{\cosh(2\xi y)\sin(2\xi y)\cosh(\xi)\sin(\xi)}{\xi(\sinh(\xi)\cosh(\xi)+\sin(\xi)\cos(\xi))}\right] \quad (35)$$

where

$$\xi = \frac{1}{2}[Ra\cos(\theta)]^{1/4} \text{ for } \theta<90° \quad (36a)$$

$$\xi = \frac{1}{2\sqrt{2}}[-Ra\cos(\theta)]^{1/4} \text{ for } \theta>90° \quad (36b)$$

Interestingly, some fundamental insights into this kind of solutions can be obtained (as noticed by Delgado-Buscalioni and Crespo del Arco, 2001) considering the vorticity-production term of the vorticity balance equation. For the present case such term reads:

$$Ra\left(\frac{\partial T}{\partial x}\sin(\theta)+\frac{\partial T}{\partial y}\cos(\theta)\right) \quad (37)$$

The evolution of these flow profiles with Ra is ruled by the balance of dissipation and production of vorticity by buoyant forces. According to eq. (37) above, the production of vorticity due to the y and x components of buoyancy are proportional to Ra $\sin(\theta)\partial T/\partial x$ and Ra $\cos(\theta)\partial T/\partial y$, respectively.

The temperature y-gradient is created by the flow advection and at low enough values of Ra it is negligibly small; therefore, at small Ra and for any (not vertical) inclination the flow is generated solely by the y component of buoyancy (at low values of Ra, in the conducting regime the cross-stream temperature gradient is vanishingly small and the vorticity is generated by the cross-stream component of gravity, at a rate given by $\sin(\theta)\partial T/\partial x$; this induces a cellular flow whose y-dependence coincides for θ=90° with the profile (6)).

As Ra increases, the streamwise advection creates an increasing (positive) temperature gradient along the y axis, which acts as another source of motion owing to the presence of the streamwise component of buoyancy; as explained before, this term produces vorticity at a rate given by $\cos(\theta)\partial T/\partial y$ and hence its effect greatly depends on the range of the inclination angle.

When heating from above (θ>90°) the effect of the axial buoyancy is to suppress the convection in the center part of the layer, as long as $\cos(\theta)\partial T/\partial y>0$ while $\sin(\theta)\partial T/\partial x<0$. For large enough Ra and θ>90°, the flow is confined to small regions near the walls where $\partial T/\partial y\cong 0$.

On the contrary, if the cavity is heated from below (θ<90°), both sources of vorticity have the same sign and as Ra increases a positive feedback loop between u(y) and T(y) occurs: any increment of the flow intensity increases the cross-stream temperature gradient, which in turn, enhances the intensity of the flow.

For conducting boundaries the expressions for heating from below and from above read:

Heating from below (0°<θ<90°)

$$u = \frac{Ra\sin(\theta)}{16}\left[\frac{\sinh(2\xi y)\sin(\xi)-\sinh(\xi)\sin(2\xi y)}{\xi^2(\sinh(\xi)\sin(\xi))}\right] \quad (38)$$

$$T = x+\frac{1}{2}\tan(\theta)\left[2y-\frac{1}{2}\frac{\sinh(2\xi y)\sin(\xi)+\sinh(\xi)\sin(2\xi y)}{\sinh(\xi)\sin(\xi)}\right] \quad (39)$$

Heating from above (90°<θ<180°)

$$u(y) = \frac{Ra\sin(\theta)}{16}\left[\frac{\cosh(2\xi y)\sin(2\xi y)\sinh(\xi)\cos(\xi)}{\xi^2(\sinh^2(\xi)\cos^2(\xi)+\cosh^2(\xi)\sin^2(\xi))}-\frac{\sinh(2\xi y)\cos(2\xi y)\cosh(\xi)\sin(\xi)}{\xi^2(\sinh^2(\xi)\cos^2(\xi)+\cosh^2(\xi)\sin^2(\xi))}\right] \quad (40)$$

$$T = x+\frac{1}{2}\tan(\theta)\left[2y-\frac{\sinh(2\xi y)\cos(2\xi y)\sinh(\xi)\cos(\xi)}{(\sinh^2(\xi)\cos^2(\xi)+\cosh^2(\xi)\sin^2(\xi))}-\frac{\cosh(2\xi y)\sin(2\xi y)\cosh(\xi)\sin(\xi)}{(\sinh^2(\xi)\cos^2(\xi)+\cosh^2(\xi)\sin^2(\xi))}\right] \quad (41)$$

with ξ given by eqs. (36).

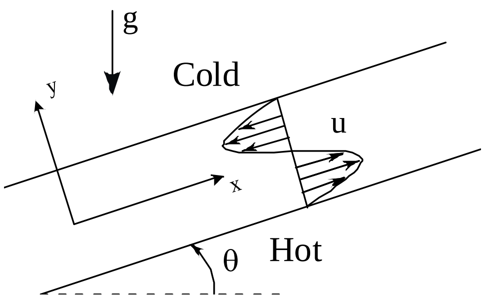


FIG. 7 SKETCH OF A LAYER OF INFINITE EXTENT INCLINED WITH RESPECT TO THE HORIZONTAL DIRECTION WITH HEATING APPLIED THROUGH THE BOTTOM WALL (TEMPERATURE GRADIENT IMPOSED ALONG THE y DIRECTION).

Equations (32)-(35) and (38)-(41) provide analytical solutions if the imposed ΔT is parallel to the walls as shown in Fig. 6; thermogravitational convection in an inclined layer, however, also admits exact solutions if the imposed T is primarily perpendicular to the walls i.e. if such temperature gradient acts across the thickness of the layer (i.e. layer with upper and lower walls kept at uniform different temperatures as shown in Fig. 7). Such a solution reads:

$$u = \frac{Ra}{6}\sin(\theta)\left(y^3 - \frac{y}{4}\right) \tag{42}$$

with $Ra=g\beta\Delta Td^3/\nu\alpha$.

The corresponding distribution of temperature along y is governed by diffusion only and is approximately uniform throughout the plane of the layer.

Velocity (u) tends to zero (as expected) in the limit as $\theta\to 0$ (in such a case, in fact, it is a well-known fact that convection arises only if a given threshold of the Rayleigh number is exceeded).

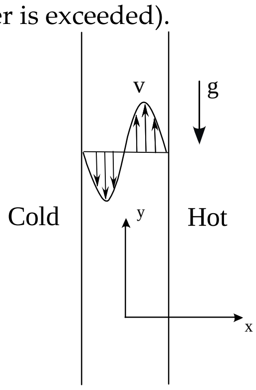


FIG. 8 SKETCH OF A TRANSVERSELY HEATED VERTICAL CAVITY IN THE LIMIT AS THE VERTICAL SCALE OF MOTION TENDS TO INFINITY.

It is also worth noticing that for θ=90° one recovers the idealized case of a transversely heated vertical cavity (in the limit as the vertical scale of motion tends to infinity, see Fig. 8):

$$v = \frac{Ra}{6}\left(x^3 - \frac{x}{4}\right) \tag{43}$$

this profile is generally referred to as "conduction-regime" solution (owing to the associated temperature profile along x, that as mentioned before is linear).

Convection with Vibrations

Pure Thermovibrational Flow

Let us recall that thermovibrational convection can be regarded as a "variant" of the standard thermogravitational convection for which the steady Earth gravity acceleration is replaced by an acceleration oscillating in time with a given frequency (Lappa, 2004, 2010).

Disturbances induced in a fluid by a sinusoidal displacement of the related container along a given direction ($\underline{\hat{n}}$ is the related unit vector)

$$\underline{s}(t) = \mathrm{b}\sin(\omega\ \mathrm{t})\underline{\hat{n}} \tag{44}$$

where b is the amplitude and ω=2πf (f is the frequency) induce an acceleration:

$$\underline{g}(t) = \underline{g}_{\omega}\ \sin(\omega\ \mathrm{t}) \tag{45}$$

where $\underline{g}_{\omega} = b\,\omega^2\underline{\hat{n}}$; which means vibrating a system with frequency f and displacement amplitude b corresponds to a sinusoidal gravity modulation with the same frequency and acceleration amplitude $b\omega^2$ and vice versa (accordingly, hereafter the terms "gravity modulation", "periodic acceleration", "container vibration" and g-jitter will be used as synonyms). This also means that $Ra_{\omega} = \frac{b\omega^2\beta_T\Delta Td^3}{\nu\alpha}$ can be regarded as a variant of the classical Rayleigh number with the steady acceleration being replaced by the amplitude of the considered periodic acceleration.

In general, the treatment of the problem is possible in terms of three independent nondimensional parameters only, where the first is the well-known Prandtl number (Pr) and the others are the nondimensional frequency (Ω) and displacement (Λ), defined as:

$$\Omega = \frac{\omega d^2}{\alpha} \tag{46}$$

$$\Lambda = b\frac{\beta_T\Delta T}{d} \tag{47}$$

where, obviously $\Lambda\Omega^2 = PrRa_{\omega}$

In the specific case of sufficiently small amplitudes (Λ<<1) and sufficiently large frequencies (Ω>>1) of the

vibrations, it is known (see the historical background reported in Lappa, 2010) that, for given Prandtl number, vibrational convection depends only on one relevant dimensionless parameter, the socalled vibrational Rayleigh number originally introduced by Gershuni (Gershuni and Zhukhovitskii, 1979):

$$Ra_V = \frac{(b\omega\beta_T \Delta T d)^2}{2\nu\alpha} = \frac{(\beta_T \Delta T d)^2}{2\nu\alpha}\left(\frac{g_\omega}{\omega}\right)^2 = \frac{\Omega^2\Lambda^2}{2\Pr} \tag{48}$$

where ΔT is the considered applied temperature gradient.

Under the assumptions of small amplitudes (Λ<<1) and large frequencies of the vibrations (Ω>>1), the Gershuni's formulation leads to a closed set of equations for the time-averaged quantities. The time-averaged continuity and energy equations remain formally unchanged (i.e. they correspond to eqs. (1), and (3), respectively, with $T = \overline{T}$ and $\underline{V} = \overline{\underline{V}}$); the time-averaged momentum equation must be re-written as:

$$\frac{\partial \overline{\underline{V}}}{\partial t} = -\underline{\nabla}\overline{p} - \underline{\nabla}\cdot\left[\overline{\underline{V}}\,\overline{\underline{V}}\right] + \Pr\nabla^2\overline{\underline{V}} + \Pr Ra_V\left[(\underline{w}\cdot\underline{\nabla}\overline{T})\underline{\hat{n}} - \underline{w}\cdot\underline{\nabla}\underline{w}\right] \tag{49}$$

where $\underline{w}$ appearing in the production term (the mathematical details related to the derivation of this term are given in Lappa, 2010) is an auxiliary potential function satisfying the equations:

$$\underline{\nabla}\cdot\underline{w} = 0 \tag{50a}$$

$$\underline{\nabla}\wedge\underline{w} = \underline{\nabla}\overline{T}\wedge\underline{\hat{n}} \rightarrow \nabla^2\underline{w} = -\underline{\nabla}\wedge\left(\underline{\nabla}\overline{T}\wedge\underline{\hat{n}}\right) \tag{50b}$$

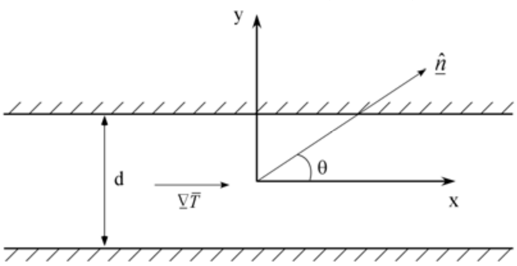


FIG. 9 SKETCH OF LAYER OF INFINITE EXTENT SUBJECTED TO VIBRATIONS.

Following Birikh (1990), the mathematical problem related to the derivation of an analytic solution for the conditions corresponding to Fig. 9 can be defined as follows.

Assuming a generic plane-parallel flow solution in the form:

$$\underline{V} = \begin{bmatrix} u(y) \\ 0 \\ 0 \end{bmatrix} \tag{51}$$

$$T = x + f(y) \tag{52}$$

$$\underline{w} = \begin{bmatrix} w(y) \\ 0 \\ 0 \end{bmatrix} \tag{53}$$

the original partial differential equations for momentum, energy and the auxiliary potential function $\underline{w}$ given before can be reduced to a system of ordinary differential equations (here θ is the angle between the shaking direction and the x-axis):

$$\frac{d^3u}{dy^3} + Ra_v\frac{dw}{dy}\cos(\theta) = 0 \tag{54}$$

$$\frac{d^2 f}{dy^2} + u = 0 \tag{55}$$

$$\frac{dw}{dy} = \frac{df}{dy}\cos(\theta) - \sin(\theta) \tag{56}$$

that combining eq. (54) with eq. (56) can be cast in compact form as:

$$\frac{d^3u}{dy^3} + Ra_v\frac{df}{dy}\cos^2(\theta) = Ra_v\cos(\theta)\sin(\theta) \tag{57}$$

$$\frac{d^2 f}{dy^2} + u = 0 \tag{58}$$

with $Ra_V = \frac{(b\omega\beta_T\gamma d^2)^2}{2\nu\alpha}$ where γ is the rate of uniform temperature increase along the x axis and boundary conditions at y=±1/2: u=0 (no-slip) and df/dy=0 or f=0 for adiabatic or conducting walls, respectively.

Notably, the dimensionless characteristic number of the problem together with the components of the unit vector $\underline{\hat{n}}$ appearing in these equations can be grouped in two parameters only:

$$R_1 = Ra_v\cos(\theta)\sin(\theta) \tag{59}$$

$$R_2 = Ra_v\cos^2(\theta) \tag{60}$$

of which, the first one can be used to obtain conditions corresponding to the existence of the socalled "states of quasi-equilibrium" (no time-average flow) by simply setting it equal to zero (which leads to two possible cases of quasi equilibrium: the well-known condition of vibrations parallel to the temperature gradient (θ=0°→sin(θ)=0→R_1=0) and vibrations perpendicular to the temperature gradient (θ=90°→cos(θ)=0→ R_1=0)).

In general, the analytic form of the plane-parallel flow established in the layer for other values of θ not corresponding to equilibrium (θ≠0°, θ≠90°) must be determined by solution of the system (54)-(56) with the additional constraints:

$$\int_{-1/2}^{1/2} u(y)\,dy = 0 \quad \text{and} \quad \int_{-1/2}^{1/2} w(y)\,dy = 0 \tag{61}$$

As illustrated in detail by Birikh (1990), this approach leads to the following mathematical expressions:

$$u(y) = -\frac{R_1}{16}\left[\frac{\cosh(2\xi y)\sin(2\xi y)\sinh(\xi)\cos(\xi))}{\xi^3(\sinh(\xi)\cosh(\xi)+\sin(\xi)\cos(\xi))} - \frac{\sinh(2\xi y)\cos(2\xi y)\cosh(\xi)\sin(\xi)}{\xi^3(\sinh(\xi)\cosh(\xi)+\sin(\xi)\cos(\xi))}\right] \quad (62)$$

$$f(y) = \frac{1}{2}\tan(\theta)\left[2y - \frac{\sinh(2\xi y)\cos(2\xi y)\sinh(\xi)\cos(\xi)}{\xi(\sinh(\xi)\cosh(\xi)+\sin(\xi)\cos(\xi))} - \frac{\cosh(2\xi y)\sin(2\xi y)\cosh(\xi)\sin(\xi)}{\xi(\sinh(\xi)\cosh(\xi)+\sin(\xi)\cos(\xi))}\right] \quad (63)$$

for the case with adiabatic walls

$$u(y) = -\frac{R_1}{16}\left[\frac{\cosh(2\xi y)\sin(2\xi y)\sinh(\xi)\cos(\xi)}{\xi^2(\sinh^2(\xi)\cos^2(\xi)+\cosh^2(\xi)\sin^2(\xi))} - \frac{\sinh(2\xi y)\cos(2\xi y)\cosh(\xi)\sin(\xi)}{\xi^2(\sinh^2(\xi)\cos^2(\xi)+\cosh^2(\xi)\sin^2(\xi))}\right] \quad (64)$$

$$f(y) = \frac{1}{2}\tan(\theta)\left[2y - \frac{\sinh(2\xi y)\cos(2\xi y)\sinh(\xi)\cos(\xi))}{(\sinh^2(\xi)\cos^2(\xi)+\cosh^2(\xi)\sin^2(\xi))} - \frac{\cosh(2\xi y)\sin(2\xi y)\cosh(\xi)\sin(\xi)}{(\sinh^2(\xi)\cos^2(\xi)+\cosh^2(\xi)\sin^2(\xi))}\right] \quad (65)$$

for the case with conducting walls
where

$$\xi = \frac{1}{2\sqrt{2}}\left[Ra_v \cos^2(\theta)\right]^{1/4} \quad (66)$$

Modulated Buoyant Flows

There have also been studies on the effect of time-modulated gravity (or vibrations) on systems with basic buoyant convection induced by vertical static (steady and uniform) gravity and horizontal temperature gradients

The simplest model for this kind of flows is the canonical layer of infinite horizontal extent already examined before for pure buoyancy (the Hadley flow) and vibrational flow (pure thermovibrational flow).

By analogy with the analogous problem treated for the pure thermovibrational case, let us start the related discussion by observing that under the effect of high-frequency vibrations (in the framework of the averaged formulation discussed before) also this system admits solutions corresponding to quasi-equilibrium, i.e. states in which the mean velocity is zero. These states are made possible by a perfect balance of the static component of the body force and pressure gradient established in the liquid. The related mathematical (necessary) conditions of existence for the specific case considered here, i.e. vertical steady gravity and horizontal temperature gradients $\underline{\nabla}\bar{T}_o = \underline{i}_x$, reduce to:

$$Ra_v \cos(\theta)\sin(\theta) - Ra = 0 \quad (67)$$

When requisites for mechanical quasi-equilibrium are not satisfied time-average fluid motion arises. Such convective states are twodimensional and admit analytic solution in the form of plane-parallel flow. Following the general concepts introduced before for pure thermovibrational flows, these solutions can be represented as

$$u(y) = -\frac{R_1}{16}\left[\frac{\cosh(2\xi y)\sin(2\xi y)\sinh(\xi)\cos(\xi)}{\xi^3(\sinh(\xi)\cosh(\xi)+\sin(\xi)\cos(\xi))} - \frac{\sinh(2\xi y)\cos(2\xi y)\cosh(\xi)\sin(\xi)}{\xi^3(\sinh(\xi)\cosh(\xi)+\sin(\xi)\cos(\xi))}\right] \quad (68)$$

$$f(y) = \frac{1}{2}\frac{R_1}{R_2}\left[2y - \frac{\sinh(2\xi y)\cos(2\xi y)\sinh(\xi)\cos(\xi)}{\xi(\sinh(\xi)\cosh(\xi)+\sin(\xi)\cos(\xi))} - \frac{\cosh(2\xi y)\sin(2\xi y)\cosh(\xi)\sin(\xi)}{\xi(\sinh(\xi)\cosh(\xi)+\sin(\xi)\cos(\xi))}\right] \quad (69)$$

for the case with adiabatic walls

$$u(y) = -\frac{R_1}{16}\left[\frac{\cosh(2\xi y)\sin(2\xi y)\sinh(\xi)\cos(\xi))}{\xi^2(\sinh^2(\xi)\cos^2(\xi)+\cosh^2(\xi)\sin^2(\xi))} - \frac{\sinh(2\xi y)\cos(2\xi y)\cosh(\xi)\sin(\xi)}{\xi^2(\sinh^2(\xi)\cos^2(\xi)+\cosh^2(\xi)\sin^2(\xi))}\right] \quad (70)$$

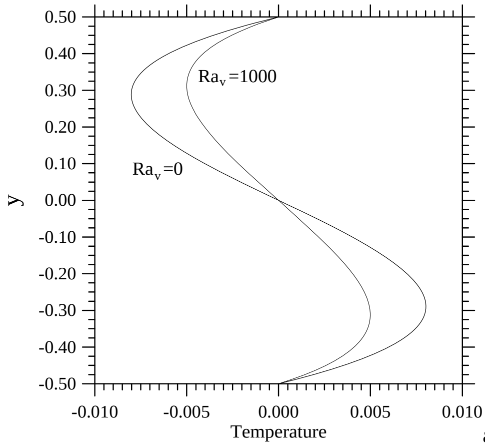
$$f(y) = \frac{1}{2}\frac{R_1}{R_2}\left[2y - \frac{\sinh(2\xi y)\cos(2\xi y)\sinh(\xi)\cos(\xi)}{(\sinh^2(\xi)\cos^2(\xi)+\cosh^2(\xi)\sin^2(\xi))} - \frac{\cosh(2\xi y)\sin(2\xi y)\cosh(\xi)\sin(\xi)}{(\sinh^2(\xi)\cos^2(\xi)+\cosh^2(\xi)\sin^2(\xi))}\right] \quad (71)$$

for the case with conducting walls (see Fig. 10)
where

$$\xi = \frac{1}{2\sqrt{2}}\left[R_2\right]^{1/4} \quad (72)$$

$$R_1 = Ra_v \cos(\theta)\sin(\theta) - Ra \quad (73)$$

$$R_2 = Ra_v \cos^2(\theta) \quad (74)$$



a)

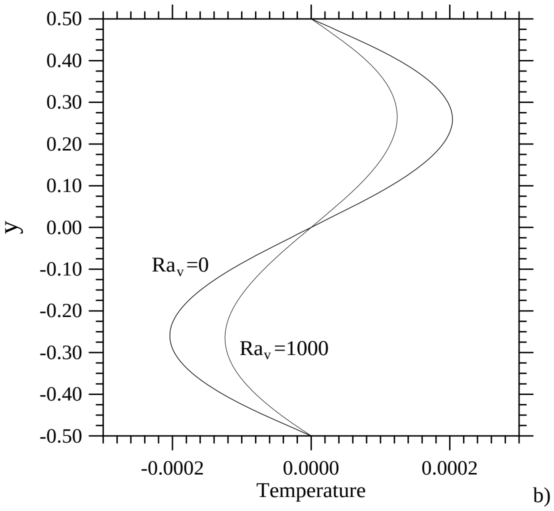


FIG. 10 EXACT SOLUTION FOR THE CASE OF MIXED THERMOGRAVITATIONAL-THERMOVIBRATIONAL (AVERAGE) CONVECTION IN INFINITE LAYER WITH TOP AND BOTTOM SOLID CONDUCTING WALLS (θ=0°, Ra=1): (a) VELOCITY PROFILE u(y); (b) TEMPERATURE PROFILE f(y).

Mixed Marangoni/Thermovibrational Convection

Like buoyancy flow, also the thermal Marangoni flow can be strongly affected by imposed vibrations. Such a case is considered in this section (surface-tension-induced flow interacting with vibrational effects in the absence of a static acceleration component, i.e. Ma≠0, Ra_ω or Ra_v≠0 and Ra=0).

Interesting results along these lines deserving attention are due to Suresh and Homsy (2001), who considered the infinite parallel Marangoni flow subjected to gravitational modulation at low frequencies (where the averaged model is not applicable) for the case in which finite-frequency vibrations are perpendicular to the layer (θ=90°), (Fig. 11; they assumed as base unmodulated Marangoni flow the popular "return flow solution" given by eq. (20) and employed a quasi-steady approach, in the limit of *very low forcing frequency*).

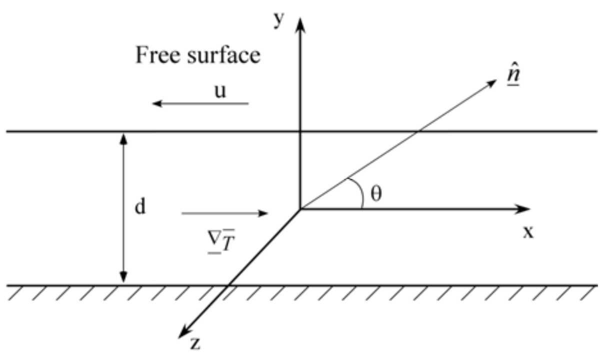


FIG. 11 SKETCH OF FLUID LAYER OF INFINITE EXTENT WITH UPPER FREE SURFACE SUBJECTED TO VIBRATIONS $\underline{s}(t) = b\sin(\omega t)\underline{\hat{n}}$ WITH ARBITRARY DIRECTION.

Following the general concepts introduced earlier, such solution (see Figs. 12) can be expressed as the superposition of two components, i.e. the steady Marangoni component proportional to Ma ($\sigma_T\gamma d^2/\mu\alpha$) and a periodic vibrational component proportional to Ra_ω ($b\omega^2\beta_T\gamma d^4/\nu\alpha=\Lambda\Omega^2/Pr$, where γ is the rate of uniform temperature increase along the x axis):

$$u = Ma g_M(y) + Ra_\omega g_B(y)\exp(i\Omega t) \quad (75)$$

$$T = x + Ma f_M(y) + Ra_\omega f_B(y)\exp(i\Omega t) \quad (76)$$

where, as already reported in the case of pure Marangoni flow, for adiabatic horizontal walls:

$$g_M(y) = -\frac{1}{4}\left(3y^2 + y - \frac{1}{4}\right) \quad (77)$$

$$f_M(y) = -\frac{1}{48}\left(3y^4 + 2y^3 - \frac{3}{2}y^2 - \frac{3}{2}y - \frac{5}{16}\right) \quad (78)$$

and, as analytically determined by Suresh and Homsy (2001):

$$g_B(y) = mc_1\exp(m\eta) - mc_2\exp(-m\eta) + c_3 - \frac{\eta}{m^2} \quad (79)$$

and:

$$f_B(y) = d_1\exp(n\eta) + d_2\exp(-n\eta) + d_3\exp(m\eta) + d_4\exp(-m\eta) + d_5 - \eta/m^2n^2 \quad \text{for Pr}\neq 1 \quad (80a)$$

$$f_B(y) = (d_1 + d_3\eta)\exp(m\eta) + (d_2 + d_4\eta)\exp(-m\eta) + d_5 - \eta/m^4 \quad \text{for Pr}=1 \quad (80b)$$

with

$$\eta = \left(y + \frac{1}{2}\right) \quad (81a)$$

$$m = \left(\frac{i\Omega}{Pr}\right)^{1/2} \quad (81b)$$

$$n = (i\Omega)^{1/2} \quad (81c)$$

$$c_1 = \frac{\exp(-m)(m^2/2 - 1) - m + 1}{2m^4(-m\cosh(m) + \sinh(m))} \quad (82a)$$

$$c_2 = \frac{\exp(m)(-m^2/2 + 1) - m - 1}{2m^4(-m\cosh(m) + \sinh(m))} \quad (82b)$$

$$c_3 = m(c_2 - c_1) \quad (82c)$$

$$c_4 = -(c_1 + c_2) \quad (82d)$$

if Pr≠1

$$d_1 = \frac{1}{2n\sinh(n)}\left\{\frac{m^2}{n^2 - m^2}\left[(c_1 + c_2)\exp(-n) - c_1\exp(m) - c_2\exp(-m)\right] + \frac{1 - \exp(-n)}{m^2n^2}\right\} \quad (83a)$$

$$d_2 = \frac{1}{2n\sinh(n)}\left\{\frac{m^2}{n^2 - m^2}\left[(c_1 + c_2)\exp(n) - c_1\exp(m) - c_2\exp(-m)\right] + \frac{1 - \exp(n)}{m^2n^2}\right\} \quad (83b)$$

$$d_3 = \frac{mc_1}{n^2 - m^2} \tag{83c}$$

$$d_4 = \frac{-mc_2}{n^2 - m^2} \tag{83d}$$

$$d_5 = \frac{c_3}{n^2} \tag{83e}$$

if Pr=1

$$d_1 = \frac{1}{4m\sinh(m)}\Big[2\sinh(m)c_1 + m(c_1\exp(m) - c_2\exp(-m)) + \frac{2(1-\exp(-m))}{m^4}\Big] \tag{84a}$$

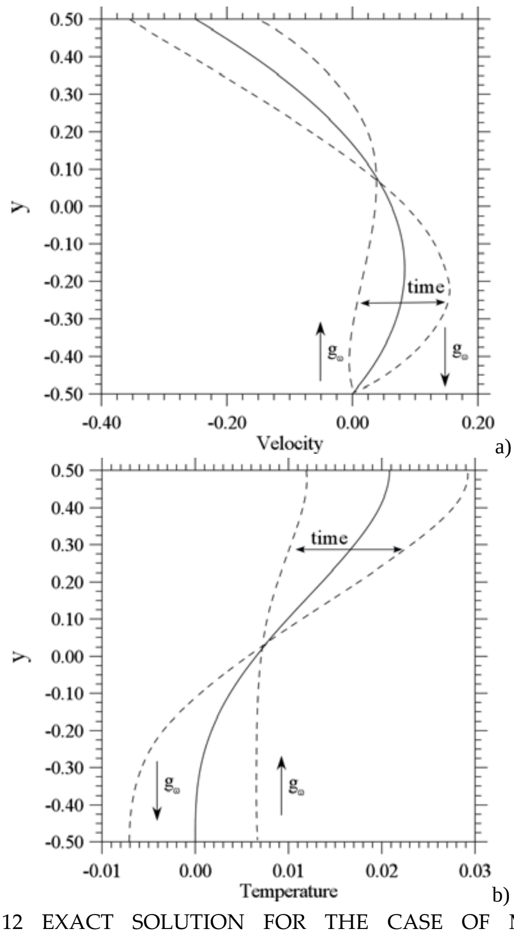
$$d_2 = \frac{1}{4m\sinh(m)}\Big[-2\sinh(m)c_2 + m(c_1\exp(m) - c_2\exp(-m)) + \frac{2(1-\exp(m))}{m^4}\Big] \tag{84b}$$

$$d_3 = -\frac{c_1}{2} \tag{84c}$$

$$d_4 = -\frac{c_2}{2} \tag{84d}$$

$$d_5 = \frac{c_3}{m^2} \tag{84e}$$

with the horizontal boundaries located at y=±1/2.



FIGS. 12 EXACT SOLUTION FOR THE CASE OF MIXED MARANGONI-THERMOVIBRATIONAL CONVECTION IN INFINITE LAYER WITH ADIABATIC BOUNDARIES (Pr=1, Ma=1, θ=90°, Ra_ω=5, Ω=1).

Solutions in the Presence of Static and Uniform Magnetic Fields

In many material processing techniques, a magnetic field is commonly used to control the liquid flows. Its action can lead to the braking of the flow (i.e. the reduction of the rate of convective transport) or to the damping of possible oscillatory convective instabilities. For these reasons, a number of theoretical studies have appeared during recent years concerning the effect of magnetic fields on the basic flow motion in several geometrical models of semiconductor growth techniques.

The present section illustrates the effect of a constant magnetic field (static and uniform) on the fundamental solutions for gravitational and Marangoni convection introduced in earlier sections.

Physical Principles and Governing Equations

The motion of an electrically conducting melt under a magnetic field induces electric currents. Lorentz forces, resulting from the interaction between the electric currents and the magnetic field, affect the flow.

In the following, as stated in the title of the present subsection, the "physics" of the problem and the introduction of the corresponding model equations are considered for the simple and representative case of constant magnetic fields.

A uniform magnetic field with magnetic flux density (also referred to as magnetic induction) Bo generates a damping Lorentz force through the aforementioned electric currents induced by the motion across the magnetic field that in dimensional form can be expressed as $\underline{F}_L = \underline{J}_f \wedge \underline{B}$ where $\underline{J}_f$ is the electrical current density. Like other body forces (e.g., the buoyancy forces), an account for this force can be yielded simply adding a relevant term to the momentum equation.

Following this approach, scaling the magnetic flux density with B_o and the electric current density $\underline{J}_f$ with $\sigma_e V_{ref} B_o$, the momentum equation including the Lorentz force, can be written in nondimensional form (assuming as usual Vref=α/d) and in the absence of phase transitions as:

$$\frac{\partial \underline{V}}{\partial t} = -\underline{\nabla} p - \underline{\nabla}\cdot[\underline{V}\underline{V}] + \Pr\nabla^2\underline{V} - \Pr RaT\underline{i}_g + \Pr Ha^2\left(\underline{J}_f \wedge \underline{i}_{Bo}\right) \tag{85}$$

where Ha is the socalled *Hartmann number*

$$Ha = B_o d\left(\frac{\sigma_e}{\rho\nu}\right)^{1/2} \quad (86)$$

σ_e is the electrical conductivity and iBo the unit vector in the direction of B_o.

The nondimensional electric current density $\underline{J}_f$ can be expressed via the Ohm's law for a moving fluid as:

$$\underline{J}_f = \underline{E} + \underline{V} \wedge \underline{i}_{Bo} \quad (87)$$

where $\underline{E}$ is the electric field normalized by $V_{ref}B_o$. Since (see Lappa, 2010), in many cases of technological interest) the unsteady induced field is negligible, in particular, the electric field can be written as the gradient of an electric potential:

$$\underline{E} = -\underline{\nabla}\Phi_e \quad (88)$$

The conservation of the electric current density gives:

$$\underline{\nabla} \cdot \underline{J}_f = 0 \quad (89)$$

which combined with eq. (88) leads to a Poisson equation for the electric potential:

$$\nabla^2\Phi_e = \underline{\nabla} \cdot \left(\underline{V} \wedge \underline{i}_{Bo}\right) = \underline{i}_{Bo} \cdot \underline{\nabla} \wedge \underline{V} \quad (90)$$

Finally, the momentum equation can be rewritten as:

$$\frac{\partial \underline{V}}{\partial t} = -\underline{\nabla}p - \underline{\nabla} \cdot [\underline{VV}] + \Pr\nabla^2\underline{V} - \Pr RaT\underline{i}_g + \Pr Ha^2\left(-\underline{\nabla}\Phi_e + \underline{V} \wedge \underline{i}_{Bo}\right) \wedge \underline{i}_{Bo} \quad (91)$$

Buoyant Convection with a Magnetic Field

The simplest model in such a context is the flow in an infinite planar liquid metal layer confined between two horizontal solid walls driven by a horizontal temperature gradient (the canonical infinite Hadley flow whose general properties have been treated in a previous section in the absence of magnetic fields).

Notably, for electrically insulating horizontal walls such a flow admits analytical expression even in the cases in which it is subjected to a magnetic field with direction vertical or parallel to the applied temperature gradient, or even directed spanwise to the basic flow (see Fig. 13).

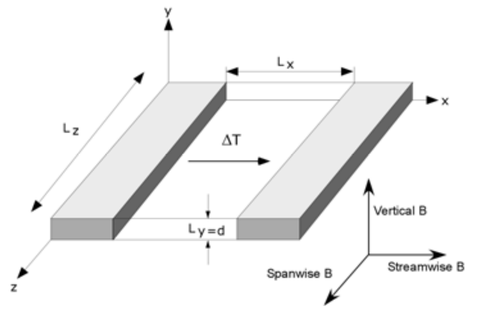


FIG. 13 SKETCH OF LIQUID LAYER UNDER THE EFFECT OF A HORIZONTAL TEMPERATURE GRADIENT AND A MAGNETIC FIELD WITH VERTICAL, STREAMWISE OR SPANWISE DIRECTION.

In the case of vertical field, the analytic solution (see, e.g., Kaddeche et al., 2003) reads:

$$u = \frac{Ra}{Ha^2}\left[\frac{\sinh(Hay)}{2\sinh(Ha/2)} - y\right] \quad (92)$$

$$T = x + \frac{Ra}{Ha^2}\left[\frac{1}{2Ha^2}\frac{\sinh(Hay)}{\sinh(Ha/2)} - \frac{y^3}{6} + \eta y\right] \quad (93)$$

with the horizontal boundaries located at y=±1/2 and the parameter η given by:

$$\eta = \frac{1}{8} - \frac{\cosh(Ha/2)}{2Ha\sinh(Ha/2)} \quad (94a)$$

for adiabatic boundaries and

$$\eta = \frac{1}{24} - \frac{1}{Ha^2} \quad (94b)$$

for conducting boundaries.

If a horizontal magnetic field parallel to the applied temperature gradient (magnetic field applied along the x-direction) is considered, there is no direct effect of the field on the parallel flow in the layer as the velocity is parallel to the field direction ($\underline{V} \wedge \underline{i}_{Bo} = 0$), which (taking into account eqs. (87)-(90)) leads to the vanishing of the production term in eq. (91). The basic flow is, hence, the flow without magnetic field given by eq. (18).

For a magnetic field still horizontal but directed spanwise to the basic flow (i.e. a magnetic field applied along the z-direction), the basic parallel flow in the layer is still the flow without magnetic field.

A justification for such a behavior is not straightforward as one would assume, and deserves some specific additional insights.

First of all, it has to be noted that as long as we consider the basic flow as being horizontally homogeneous, the electric field induced by the liquid motion is uniform and directed perpendicular to the plane of the layer. Zero circulation of this field (related to the fundamental irrotational nature of electric fields when steady magnetic fields are considered) implies that no electric currents closing within the liquid layer can be induced, but the electric impermeability of the horizontal boundaries also precludes the possibility of any electric current passing normally through the layer; thereby, insulating horizontal walls lead to a separation of electric charges over the depth of the layer, so giving rise to an electrostatic field which

cancels that induced by the liquid motion. As a result, there is no electric current induced by a horizontally uniform basic flow in a coplanar magnetic field and, therefore, there is no direct influence of the magnetic field on the basic flow.

Marangoni Flow with a Magnetic Field

Notably, like the Hadley flow discussed before, also the infinite surface-tension-driven flow admits analytical expression under the effect of a magnetic field.

In particular, if the magnetic field is parallel to the free surface, i.e. satisfies $\underline{B}\cdot\underline{i}_y=0$ (hereafter referred to as coplanar field), then, following the same arguments already provided for the infinite Hadley flow, it can be concluded that the field has no influence on the basic flow, and consequently the parallel flow remains the same as without it (this is exactly the return flow given by eq. (20)).

By contrast, if the magnetic field is perpendicular to the free surface the analytic solution (neglecting exponentially small terms for strong magnetic field Ha>>1) becomes (Priede et al., 1995):

$$u=-Ma\left\{\frac{1}{Ha}\exp\left[-\left(\frac{1}{2}-y\right)Ha\right]+\frac{1}{Ha^2}\exp\left[-\left(\frac{1}{2}+y\right)Ha\right]-\frac{1}{Ha^2}\right\} \tag{95}$$

$$T=x-Ma\left\{-\frac{y}{Ha^2}\left(\frac{y+1}{2}\right)+\frac{1}{Ha^3}\exp\left[-\left(\frac{1}{2}-y\right)Ha\right]+\frac{y}{Ha^3}+\frac{1}{Ha^4}\exp\left[-\left(\frac{1}{2}+y\right)Ha\right]\right\} \tag{96}$$

where, as usual, the horizontal boundaries are assumed to be located at $y=\pm 1/2$ and $Ma=\sigma_T\gamma d^2/\mu\alpha$ (γ being the rate of uniform temperature increase along the x axis and d the depth of the layer), $Ha=B_o d\,(\sigma_e/\rho\nu)^{1/2}$.

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Marcello Lappa is Senior Researcher and Activity Coordinator at Telespazio (formerly, Microgravity Advanced Research and Support Center). He has about 100 publications (most of which as single author) in the fields of fluid motion and stability behavior, organic and inorganic materials sciences and crystal growth, multiphase flows, solidification, biotechnology and biomechanics, methods of numerical analysis in computational fluid dynamics and heat/mass transfer, high performance computing (parallel machines). In 2004 he authored the book "Fluids, Materials and Microgravity: Numerical Techniques and Insights into the Physics" (2004), Elsevier Science, that owing to the topics treated is currently used as a reference book in the field of microgravity disciplines and related applications. He is Editor-in-Chief (2005-present) of the international scientific Journal "Fluid Dynamics and Materials Processing" (ISSN 1555-256X). In 2010 he authored the book "Thermal Convection: Patterns Evolution and Stability" (2010), John Wiley & Sons, in which a critical, focused and 'comparative' study of different types of thermal convection typically encountered in natural or technological contexts (thermogravitational, thermocapillary and thermovibrational) was elaborated (including the effect of magnetic fields and other means of flow control). He extended the treatment of thermal convection to the case of rotating fluids in a subsequent book ("Rotating Thermal Flows in Natural and Industrial Processes, John Wiley & Sons, 2012), expressly conceived for a broader audience of students (treating not only fluid mechanics, thermal, mechanical and materials engineering, but also meteorology, oceanography, geophysics and atmospheric phenomena in other-solar-system bodies). In recent years, he has taken care of the activities (at Telespazio) for preparation and execution of space experiments in the field of fluids onboard the International Space Station (ISS) in direct co-operation with NASA on behalf of the European Space Agency (ESA). Related activities include: design of the mission scenario (Mission Operation Implementation Concept); development of all the products required to manage the payload (procedures, payload regulations, joint operation interfaces, payload technical support concept, certification of flight readiness); coordination of experiment science teams.